Article citation info:
Khnaijar A, Benamar R. A discrete model for nonlinear vibrations of a simply supported cracked beams resting on elastic foundations. Diagnostyka. 2017;18(3):39-46

DIAGNOSTYKA, 2017, Vol. 18, No. 3

ISSN 1641-6414
e-ISSN 2449-5220

A DISCRETE MODEL FOR NONLINEAR VIBRATIONS OF A SIMPLY SUPPORTED CRACKED BEAMS RESTING ON ELASTIC FOUNDATIONS

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Abstract

The present paper introduces a discrete physical model to approach the problem of nonlinear vibrations of cracked beams resting on elastic foundations. It consists of a beam made of several small bars, evenly spaced, connected by spiral springs, presenting the beam bending stiffness. The crack is modeled by a spiral spring with a reduced stiffness and the Winkler soil stiffness is modeled using linear vertical springs. Concentrated masses, presenting the inertia of the beam, are located at the bar ends. The nonlinear effect, due to the axial forces in the bars resulting from the change in their length, is presented by longitudinal springs. This model has the advantage of simplifying parametric studies, because of its discrete nature, allowing any modification in the mass and the stiffness matrices, and in the nonlinearity tensor, to be made separately. After establishing the model, various practical applications are performed without the need of going through all the formulation again. Numerical linear and nonlinear results are given, corresponding to a cracked simply supported beam.

Keywords: nonlinear; discrete; beam; Winkler foundations; crack, vibration.

1. INTRODUCTION

Large vibration amplitudes of cracked beams resting on elastic foundations, wear a great practical and theoretical interest in civil, mechanical, and transportation engineering. In recent publications, this topic has been analyzed by combining the vibration theories used with various types of foundations modelling.

The study of the cracks behaviour involves, in addition to theoretical approaches, experimental investigations carried out in order to establish idealized crack models. The first attempts [1] to quantify local defects were those of Kirmsher [2] and Thomson [3] who simulated the effect of a notch on the structure flexibility by a local bending moment or a reduced section, with magnitudes that were experimentally estimated. Also, in the 1950s, Irwin [4, 5], Bueckner [6], Westmann and Yang [7] quantified the local flexibility of a cracked region of a structural element by relating the local flexibility to the crack stress intensity factor (SIF). Using this principle, a method was developed for the computation of the SIF based on the local bending stiffness (the inverse of the local flexibility) of a cracked rectangular beam, determined experimentally. Since 1957, several investigators computed the SIF and the local flexibility for a variety of geometries of the crack and the associated structural member. Liebowitz et al. [8-10] used existing results from fracture mechanics to calculate the local rotational flexibility of a beam of a rectangular cross-section. More recently [11], cracks have been modelled as a massless hinge with a torsional spring at the crack position. The models permitted to establish theories in the field of crack detection, based on the inverse problem. These theories were essentially based on the variation in frequencies and mode shapes induced by the presence of the crack. For the study of a beam with multiple cracks, one may use a model made of several beams connected with massless spiral springs, which may be considered a partially natural discrete model.

Ming-Hung Hsu [12] addressed the vibration analysis of an edge-cracked beam resting on elastic foundations with axial loading using the differential quadrature method. Shin Y et al. [13] investigated the effect of the crack parameters (size, location, size and number) and the foundation stiffness on the natural frequencies of an Euler-Bernouli beam for both the simply supported and the clamped beam cases. M. Attar et al. [14] studied free vibrations of a shear deformable beam with multiple open edge cracks using a lattice spring model (LSM) where the beam was discretized into a one-dimensional assembly of segments interacting via rotational and shear springs. The beam was also supported by the so-called two-parameter elastic foundation. Akbas [15] studied the free vibrations of an edge cracked functionally graded cantilever beam resting on Winkler Pasternak foundations, where the differential equations of motion were obtained using Hamilton's principle. The problem was investigated within the Euler-Bernoulli beam theory by using a finite element method. The cracked beam was modelled as an assembly of two sub-beams connected through a massless elastic rotational spring. A.C. Neves et al. [16], similarly to the work carried out in this paper, used a discrete model based on the Discrete Element Method (DEM) to study the effect of the crack on the dynamic behaviour of a cantilever beam and of a beam free of support conditions. V. Stojanović, M. D. Petković [17] studied the geometrically nonlinear free and forced vibrations of damaged high order shear deformable beams resting on a nonlinear Pasternak foundation, where they investigated the effects of the specific stiffness of the foundation on the damaged beam frequencies and displacements with the aim of equalising the response of a damaged and an intact beam.

The purpose of this paper is the development of an adaptable general discrete model for linear and nonlinear vibrations of cracked beams resting on elastic foundations, using a discrete model. The process of discretization carried out is intended to allow an efficient procedure for taking into account systematically the variation in the beam, the crack and the soil characteristics, in order to perform multiple parametric studies.

In Section 2, of the present paper, the theoretical formulation is presented and the discrete model is detailed, with a non-dimensional formulation. Section 3 is devoted to detail and discuss the results obtained for the vibration of a cracked simply supported beam resting on elastic foundations, by varying the soil and the crack parameters in both the linear and the nonlinear case.

2. GENERAL THEORY

The general theory is established in the following section for a beam, made of extensional bars, concentrated masses, spiral and linear springs, resting on elastic foundations, presented by a distribution of vertical springs, Fig. 1.

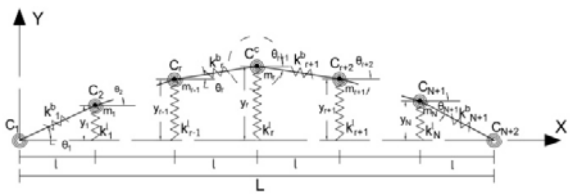


Fig. 1. The general discrete model for a cracked beam resting on elastic foundations

2.1. General formulation

Based on the model introduced by Khnaijar and Benamar [18] for nonlinear vibrations of uncracked beams resting on elastic foundations, a new cracked beam model is developed here. It consists of the *N*-degree-of-freedom discrete system shown in Fig. 1, with *N* masses $m_1,\ldots,m_N$, located at the ends of (*N*+1) rigid bars, connected by *(N+2-1)* spiral springs simulating the beam bending stiffness [19]. The crack is modeled by *a* spiral spring simulating the crack, with a reduced stiffness, as shown in Fig. 2. The stiffness of the current r^{th} spring is denoted

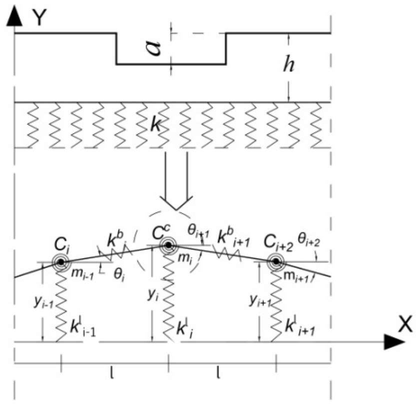


Fig. 2. Crack modelling

by C_r, for $r=1$ to $(N+2-1)$), and the stiffness of the spiral spring, presenting the crack, is denoted by C^c . The bending moment M in the r^{th} spiral spring connecting the bars (*r*-1*) and r* is given by: $M=-C_r\Delta\theta$; $\Delta\theta=\theta_r-\theta_{r-1}$ being the angle between the two bars. On the other hand, each bar is considered as a longitudinal spring of length l_r and stiffness $k^b{}_r$, $r=1$ to N. The Winkler foundations are modelled using a longitudinal vertical spring distribution, with $k^l{}_r$ presenting the stiffness coefficient of the r^{th} linear spring, for $r=1$ to N. Using vector notation $\{\mathbf{d}_1\};\{\mathbf{d}_2\};\ldots;\{\mathbf{d}_i\};\ldots;\{\mathbf{d}_N\}$ is the displacement basis, denoted as DB, defined by $\{\mathbf{d}_i\}^T=[0\ 0\ ..1...0]$ and presenting the unit displacement of the i^{th} mass. The transverse displacements of the masses m_1 to m_N from the horizontal equilibrium positions are denoted as y_1 to y_N. The non-deformed positions of the springs correspond to $y=0$. The masses displacement vector can be written in the Displacement Basis (DB) as:

$$\{\mathbf{y}\}=y_1\{\mathbf{d}_1\}+\ldots+y_i\{\mathbf{d}_i\}+\ldots y_N\{\mathbf{d}_N\} \qquad (1)$$

by considering the modal basis (MB) $\{\{\mathbf{\Phi}_1\},\{\mathbf{\Phi}_2\},\ldots,\{\mathbf{\Phi}_i\},\ldots,\{\mathbf{\Phi}_N\}\}$, in which $\{\mathbf{\Phi}_i\}^T=[\varphi_{ij}]$ is the i^{th} linear mode shape of the system obtained by solution of the linear eigenvalue problem and denoted as MB, the displacement vector can be expressed as:

$$\{\mathbf{y}\}=\overline{y_1}\{\mathbf{\Phi}_1\}+\ldots+\overline{y_i}\{\mathbf{\Phi}_i\}+\ldots+\overline{y_N}\{\mathbf{\Phi}_N\} \qquad (2)$$

The total strain energy of the beam can be written as the sum of the strain energy due to the bending denoted as V_l^s, and the strain energy induced by the Winkler springs V_l^l plus the axial strain energy V_{nl} due to the axial load induced by

Table 1. Comparison of the nondimensional fundamental frequencies of a simply supported cracked beam resting on elastic foundation for various crack positions and depths.

Crack position (c/L)	Crack depth $\xi=a/h$	Shin et al. [13][1]	λ (soil parameter)=10 Present model (Pr.) N≈10	N≈20	N≈30	Error(%)= $\frac{[13]-\text{Pr.}(N=10)}{[13]}$	Error(%)= $\frac{[13]-\text{Pr.}(N=30)}{[13]}$	Frequency drop(%)[2]
uncracked			10,3103	10,3504	10,3563			
1/8	0,02	10,3625	10,2424	10,3491	10,3550	1,16	0,07	0,01
	0,2	10,2531	10,1344	10,2398	10,2456	1,16	0,07	1,07
	0,4	9,8872	9,7711	9,8742	9,8799	1,17	0,07	4,60
	0,5	9,529	9,4132	9,5160	9,5217	1,22	0,08	8,06
1/4	0,02	10,3594	10,3058	10,3400	10,3518	0,52	0,07	0,04
	0,2	10,0003	9,9488	9,9817	9,9931	0,51	0,07	3,51
	0,4	8,9826	8,9353	8,9655	8,9759	0,53	0,07	13,33
	0,5	8,1963	8,1518	8,1803	8,1900	0,54	0,08	20,92
1/2	0,02	10,3549	10,3014	10,3389	10,3473	0,52	0,07	0,09
	0,2	9,6829	9,6334	9,6681	9,6759	0,51	0,07	6,57
	0,4	8,1405	8,0996	8,1283	8,1347	0,50	0,07	21,45
	0,5	7,1876	7,1523	7,1771	7,1826	0,49	0,07	30,64

[1]Nondimension frequency

[2]Frequency drop compared to the uncracked beam $\frac{Uncracked(N=30)-Cracked(N=30)}{Uncracked(N=30)}$

the large deflections of the beam. Thus, V_l^s, V_l^l, V_{nl} and the kinetic energy T, are as follows:

$$V_l^s=\frac{EI}{2}\int_0^L\left(\frac{\partial^2 W}{\partial x^2}\right)^2\mathrm{dx} \qquad (3)$$

$$V_l^l=\frac{k}{2}\int_0^L(W)^2\mathrm{dx} \qquad (4)$$

$$V_{nl}=\frac{ES}{8L}\left(\int_0^L\left(\frac{\partial W}{\partial x}\right)^2\mathrm{dx}\right)^2 \qquad (5)$$

$$T=\frac{\rho S}{2}\int_0^L\left(\frac{\partial W}{\partial t}\right)^2\mathrm{dx} \qquad (6)$$

where W is the beam transverse displacement. Using a generalized parameterization and the usual summation convention for repeated indices, the transverse displacement can be written as

$$W(x,t)=y_i(t)w_i(x) \text{ for } i,j=1,\dots,N \qquad (7)$$

in which w_i are the basic functions. By replacing the expression (7) for W into the energy expressions V_l^s, V_l^l, V_{nl} and T, one gets:

$$T=\frac{\overline{m_{ij}}}{2}\dot{y}_i\dot{y}_j \qquad (8)$$

$$V_l^s=\frac{\overline{k^s{}_{ij}}}{2}y_iy_j \qquad (9)$$

$$V_l^l=\frac{\overline{k^l{}_{ij}}}{2}y_iy_j \qquad (10)$$

$$V_{nl}=\frac{\overline{b_{ijkl}}}{2}y_iy_jy_ky_l \qquad (11)$$

in which $\overline{m_{ij}}$, $\overline{k^s{}_{ij}}$, $\overline{k^l{}_{ij}}$ and $\overline{b_{ijkl}}$ are defined as follows in the MB:

$$\overline{m_{ij}}=\rho S\int_0^L w_iw_j\mathrm{dx} \text{ for } i,j=1,\dots,N \qquad (12)$$

$$\overline{k^s{}_{ij}}=EI\int_0^L\frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_j}{\partial x^2}\mathrm{dx} \text{ for } i,j=1,\dots,N \qquad (13)$$

$$\overline{k^l{}_{ij}}=k\int_0^L w_iw_j\mathrm{dx} \text{ for } i,j=1,\dots,N \qquad (14)$$

$$\overline{b_{ijkl}}=\frac{ES}{4L}\int_0^L\frac{\partial w_i}{\partial x}\frac{\partial w_j}{\partial x}\mathrm{dx}\int_0^L\frac{\partial w_k}{\partial x}\frac{\partial w_l}{\partial x}\mathrm{dx} \quad \text{for } i,j,k,l=1,\dots,N \qquad (15)$$

For a conservative system, the dynamic behaviour of the structure may be obtained by Lagrange's equations:

$$-\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{y}_r}\right)+\frac{\partial T}{\partial y_r}-\frac{\partial V}{\partial y_r}=0 \text{ for } r=1,\dots,N; \qquad (16)$$

where $V=V_l^s+V_l^l+V_{nl}$. Introducing the energy expressions i.e. equations (3) to (6) into (16) gives:

$$m_{ir}\ddot{y}_i+k^s{}_{ir}y_i+k^l{}_{ir}y_i+2y_iy_jy_kb_{ijkr}=0 \quad \text{for } r=1,\dots,N \qquad (17)$$

This nonlinear differential system (17) may be expressed in a matrix form as

$$[\mathbf{M}]\{\ddot{\mathbf{y}}\}+\left(\left[\mathbf{K}^s\right]+\left[\mathbf{K}^l\right]+\frac{3}{2}\left[\mathbf{B}(\mathbf{y})\right]\right)\{\mathbf{y}\}=0 \quad (18)$$

where [**M**], [**K**^{s}], and [**B** (**y**)] are the mass, the bending rigidity and the nonlinear rigidity matrices. The beam nonlinear free response is assumed to be given by:

$$y_i = A_i^{discr}\cos(\omega^{nl}{}_{discr}t) = a_j\varphi_{ij}\cos(\omega^{nl}{}_{discr}t) \quad \text{for } i,j=1,...,N \quad (19)$$

where A_i^{discr} is the modulus of the displacement, $\omega^{nl}{}_{discr}$ is the corresponding nonlinear frequency and a_j is the modulus of the displacement y_i expressed in MB. The kinetic, linear and nonlinear strain energy expressions are obtained by replacing the expressions for y_i of equation (19) into (8) to (11), which leads to:

$$T=\frac{1}{2}a_i a_j(\omega^{nl}{}_{discr})^2\overline{m_{ij}}\sin^2(\omega^{nl}{}_{discr}t) \quad (20)$$

$$V_b^s=\frac{1}{2}a_i a_j\overline{k^s{}_{ij}}\cos^2(\omega^{nl}{}_{discr}t) \quad (21)$$

$$V_b^l=\frac{1}{2}a_i a_j\overline{k^l{}_{ij}}\cos^2(\omega^{nl}{}_{discr}t) \quad (22)$$

$$V_{nl}=\frac{1}{2}a_i a_j a_k a_l\overline{b_{ijkl}}\cos^4(\omega^{nl}{}_{discr}t) \quad (23)$$

where $\overline{m_{ij}}$, $\overline{k^s{}_{ij}}$, $\overline{k^l{}_{ij}}$ and $\overline{b_{ijkl}}$ are the MB tensor general terms given by [18,20]:

$$\overline{m_{ij}}=\varphi_{si}\varphi_{tj}m_{st} \quad (24)$$

$$\overline{k^s{}_{ij}}=\varphi_{si}\varphi_{tj}k^s{}_{st} \quad (25)$$

$$\overline{k^s{}_{ij}}=\varphi_{si}\varphi_{tj}k^s{}_{st} \quad (26)$$

$$\overline{b_{ijkl}}=\varphi_{si}\varphi_{tj}\varphi_{pk}\varphi_{ql}b_{stpq} \quad (27)$$

Replacing the expressions for T, V_l and V_{nl}, equations (20) to (23), in equation (16) and applying the harmonic balance method leads to [21,22]:

$$\left(\left[\overline{\mathbf{K}}^s\right]+\left[\overline{\mathbf{K}}^l\right]-\omega_{discr}^{nl}{}^2\left[\overline{\mathbf{M}}\right]+\frac{3}{2}\left[\overline{\mathbf{B}}(\mathbf{A})\right]\right)\{\mathbf{A}\}=0 \quad (28)$$

2.2. General expressions for the discrete model parameters

2.2.1. General expression for the dynamic tensors m_{ij}, $k^l{}_{ij}$, b_{ijkl} and ω_{nl}

The calculation of the general terms m_{ij}, $k^l{}_{ij}$, b_{ijkl} and ω_{nl} for the discrete model are detailed in [18,20]. The final expressions for m_{ij} and $k^l{}_{ij}$ are:

$$m_{ij}=\frac{\rho SL}{(N+1)}\delta_{ij} \quad \text{for } i,j=1,..,N \quad (29)$$

$$k^l{}_{ij}=k_i\frac{(N+1)}{L}\delta_{ij}=\alpha_i\frac{EI(N+1)^3}{L^3}\delta_{ij} \quad \text{for } i,j=1,..,N \quad (30)$$

α_i and λ_i being the nondimensional parameter:

$$\alpha_i=\frac{k_i L^4}{EI(N+1)^4}=\frac{\lambda_i}{(N+1)^4} \quad \text{with } \lambda_i=\frac{k_i L^4}{EI} \quad (31)$$

In the case of constant soil distribution $\alpha_i=\alpha$ and $\lambda_i=\lambda$. The fourth order tensor b_{ijkl} is given by:

$$b_{iiii}\cong 2\frac{ES}{8l^3}=2(N+1)^3\frac{ES}{8L^3} \quad \text{for } i=1,...,N \quad (32)$$

$$b_{i(i-1)(i-1)(i-1)}=b_{(i-1)i(i-1)(i-1)}=b_{(i-1)(i-1)i(i-1)}=b_{(i-1)(i-1)(i-1)i}\cong-(N+1)^3\frac{ES}{8L^3} \quad \text{for } i=2,...,N \quad (33)$$

$$b_{ii(i-1)(i-1)}=b_{(i-1)ii(i-1)}=b_{(i-1)(i-1)ii}=b_{i(i-1)(i-1)i}=b_{i(i-1)i(i-1)}=b_{(i-1)i(i-1)i}\cong(N+1)^3\frac{ES}{8L^3} \quad \text{for } i=2,...,N \quad (34)$$

$$b_{iii(i-1)}=b_{(i-1)iii}=b_{i(i-1)ii}=b_{ii(i-1)i}\cong-(N+1)^3\frac{ES}{8L^3} \quad \text{for } i=2,...,N \quad (35)$$

and the other terms of b_{ijkl} are obtained by symmetry relations, or are equal to zero. On the other hand, the spiral springs tensor is given by:

$$k^s{}_{(r-2)r}=\frac{(N+1)^2C_r}{L^2} \quad \text{for } r=3,...,N \quad (36)$$

$$k^s{}_{(r-1)r}=-\frac{2(N+1)^2}{L^2}(C_r+C_{r+1}) \quad \text{for } r=2,...,N \quad (37)$$

$$k^s{}_{rr}=\frac{(N+1)^2}{L^2}(C_r+4C_{r+1}+C_{r+2}) \quad \text{for } r=1,..,N \quad (38)$$

where:

$$C_i=\frac{EI}{l}=(N+1)\frac{EI}{L} \quad \text{with} 2\le i\le N+1 \quad (39)$$

and the other values of $k^s{}_{ij}$ are obtained by symmetry relations, or are equal to zero. The boundaries conditions of the system determine the stiffness k_{ii}^s for $i=1$ and $N+1$. For the simply supported beam examined here, the two end torsional springs C_1 and C_{N+2} have a stiffness equal to zero, since the rotation is free. Therefore $k_{ii}^s=5(N+1)^3 EI/L^3$ for $i=1$ and $N+1$. The expression for the nonlinear frequency [23] is:

Table 2: Comparison of the fundamental nondimensional nonlinear frequency parameters, in a simply supported cracked beam, for various values of crack depth and the nondimensional Winkler soil stiffness.

SIMPLY SUPPORTED BEAM (crack position *x=0,5L,* 1st mode)												
λ	0				10				100			
*A**	ξ =0	ξ =0,1	ξ =0,3	ξ =0,5	ξ =0	ξ =0,1	ξ =0,3	ξ =0,5	ξ =0	ξ =0,1	ξ =0,3	ξ =0,5
0,2	1,0056	1,0056	1,0061	1,0087	1,0051	1,0051	1,0054	1,0070	1,0028	1,0027	1,0025	1,0026
0,4	1,0223	1,0224	1,0243	1,0345	1,0202	1,0202	1,0213	1,0279	1,0110	1,0108	1,0101	1,0102
0,6	1,0494	1,0496	1,0539	1,0760	1,0449	1,0449	1,0473	1,0617	1,0247	1,0242	1,0227	1,0229
0,8	1,0863	1,0867	1,0939	1,1315	1,0785	1,0786	1,0827	1,1073	1,0434	1,0427	1,0399	1,0404
1	1,1319	1,1325	1,1434	1,1992	1,1203	1,1203	1,1266	1,1632	1,0671	1,0660	1,0617	1,0624
2	1,4577	1,4595	1,4932	1,6590	1,4213	1,4215	1,4412	1,5532	1,2469	1,2431	1,2284	1,2307
3	1,8792	1,8822	1,9407	2,2232	1,8153	1,8156	1,8503	2,0441	1,4993	1,4923	1,4647	1,4690

$$\omega_{nl}^2 = \frac{a_i a_j (\overline{k^s_{ij}} + \overline{k^l_{ij}}) + \frac{3}{2} a_i a_j a_k a_l \overline{b_{ijkl}}}{\overline{m_{ij}} a_i a_j} \quad (40)$$
$$\text{for } i, j, l, k = 1,.., N$$

2.2.2. The crack modelling

As mentioned above, the crack is modelled by an equivalent flexible spiral spring whose stiffness (inverse of the flexibility) may be estimated using the summation of the flexibility of the beam $F_i = 1/C_i$ and the crack flexibility $F_i^c = 1/C_i^c$ [14]. The total flexibility in the crack position can be expressed as:

$$F_i^{tot} = F_i + F_i^c \quad (41)$$

leading to:

$$C_i^{tot} = \frac{C_i C_i^c}{C_i + C_i^c} \quad (42)$$

The crack stiffness may be then written as:

$$C_i^c = \frac{EI}{LF^*} = \frac{(N+1)EI}{L(N+1)F^*} = \frac{C_i}{(N+1)F^*} \quad (43)$$

in which F^* is expressed according to Narkis [24] as follows:

$$F^* = 5.346h \times \left\{ \begin{array}{l} 1.8624\xi^2 - 3.95\xi^3 + 16.375\xi^4 + .. \\ -37.226\xi^5 + 76.81\xi^6 - 126.9\xi^7 + .. \\ +172\xi^8 - 143.9\xi^9 + 66.56\xi^{10} \end{array} \right\} \quad (44)$$

with $\xi = \left(\frac{a}{h}\right)$

ξ refers to the dimensionless crack depth ratio. If the crack is located at the position i, by replacing C_i^c (43) in equation (42), C_i^{tot} may be written as $C_i^{tot} = \tau_i C_i$, with:

$$\tau_i = \frac{1}{((N+1)F^* + 1)} \quad (45)$$

For an uncracked section r, $\tau_r = 1$. The generalized spiral springs tensor $k_{ij}^{s\ \text{tot}}$ terms (for the cracked and uncracked sections) may be expressed in a unified manner as

$$k^{s\ \text{tot}}_{(r-2)r} = \frac{(N+1)^2 \tau_r C_r}{L^2} \text{ for } r = 3,..., N \quad (46)$$

$$k^{s\ \text{tot}}_{(r-1)r} = -\frac{2(N+1)^2}{L^2}(\tau_r C_r + \tau_{r+1} C_{r+1}) \quad (47)$$
$$\text{for } r = 2,..., N$$

$$k^{s\ \text{tot}}_{rr} = \frac{(N+1)^2}{L^2}(\tau_r C_r + 4\tau_{r+1} C_{r+1} + \tau_{r+2} C_{r+2}) \quad (48)$$
$$\text{for } r = 1,.., N$$

For a uniform beam, by replacing C_i (39) in equations (46) to (48), one gets:

$$k^{s\ \text{tot}}_{(r-2)r} = \frac{(N+1)^3 EI}{L^3} \tau_r \text{ for } r = 3,..., N \quad (49)$$

$$k^{s\ \text{tot}}_{(r-1)r} = -\frac{2(N+1)^3 EI}{L^3}(\tau_r + \tau_{r+1}) \text{ for } r = 2,..., N \quad (50)$$

$$k^{s\ \text{tot}}_{rr} = \frac{(N+1)^3 EI}{L^3}(\tau_r + 4\tau_{r+1} + \tau_{r+2}) \quad (51)$$
$$\text{for } r = 1,.., N$$

2.3. Nondimensional formulation

To define the nondimensional parameters, put:

$$\frac{m_{ij}}{m^*_{ij}} = \frac{m_{ij}}{\overline{m^*_{ij}}} = \frac{\rho SL}{N+1} \text{ for } i, j = 1,.., N \quad (52)$$

$$\frac{k^1_{ij}}{k^{1*}_{ij}} = \frac{\overline{k^1_{ij}}}{k^{1*}_{ij}} = \frac{EI(N+1)^3}{L^3} \text{ for } i, j = 1,.., N \quad (53)$$

$$\frac{k^s_{ij}}{k^{s*}_{ij}} = \frac{\overline{k^s_{ij}}}{k^{s*}_{ij}} = \frac{EI(N+1)^3}{L^3} \text{ for } i, j = 1,.., N \quad (54)$$

$$\frac{b_{ijkl}}{b^*_{ijkl}} = \frac{\overline{b_{ijkl}}}{b^*_{ijkl}} = \frac{ES(N+1)^3}{8L^3} \text{ for } i, j, k, l = 1,.., N \quad (55)$$

$$\frac{\omega}{\omega^*} = \sqrt{EI \big/ \rho S L^4} \tag{56}$$

The nondimensional amplitude A^* is expressed as:

$$A^* = \frac{a_1}{R} \text{ with } R = \sqrt{\frac{I}{S}} \tag{57}$$

where R is the radius of gyration. By putting $K = \overline{k^s{}^*_{11}} + \overline{k^l{}^*_{11}}$, $\overline{m^*_{11}} = M$ and $\overline{b^*_{1111}} = B$, the single mode approach leads to the following expression for the nonlinear frequency parameter obtained in [23] from equation (36) and (56):

$$\omega^*_{nl}{}^2 = (N+1)^4 \frac{K}{M}\left[1 + \frac{3}{16}\frac{B}{K}\left(\frac{a_1}{R}\right)^2\right] \tag{58}$$

From the nondimensional formulation presented above, the general expressions for m^*_{ij}, k^{*s}_{ij}, k^{*l}_{ij} and b^*_{ijkl} for the discrete model become as follows:

$$m^*_{ij} = \delta_{ij} \text{ for } i, j = 1,.., N \tag{59}$$

$$k^{*l}_{ij} = \alpha_i \delta_{ij} \text{ for } i, j = 1,..., N \tag{60}$$

$$b^*_{iiii} = 2 \text{ for } i = 1,..., N \tag{61}$$

$$\begin{aligned} b^*_{i(i-1)(i-1)(i-1)} &= b^*_{(i-1)i(i-1)(i-1)} = b^*_{(i-1)(i-1)i(i-1)} \\ &= b^*_{(i-1)(i-1)(i-1)i} = -1 \text{ for } i = 2,..., N \end{aligned} \tag{62}$$

$$\begin{gathered} b^*_{ii(i-1)(i-1)} = b^*_{(i-1)ii(i-1)} = b^*_{(i-1)(i-1)ii} = \\ b^*_{i(i-1)(i-1)i} = b^*_{i(i-1)i(i-1)} = b^*_{(i-1)i(i-1)i} = 1 \\ \text{for } i = 2,..., N \end{gathered} \tag{63}$$

$$\begin{gathered} b^*_{iii(i-1)} = b^*_{(i-1)iii} = b^*_{i(i-1)ii} = b^*_{ii(i-1)i} = -1 \\ \text{for } i = 2,..., N \end{gathered} \tag{64}$$

The other values of b^*_{ijkl} are equal to zero. The general terms of the nondimensional spiral springs tensor k^{s*}_{ij} are obtained by replacing equation (54) in equations (51) to (53), are given by:

$$k^{s*\,\text{tot}}_{(r-2)r} = \tau_r \text{for } r = 3,..., N \tag{65}$$

$$k^{s*\,\text{tot}}_{(r-1)r} = -2(\tau_r + \tau_{r+1}) \text{ for } r = 2,..., N \tag{66}$$

$$\begin{gathered} k^{s*\,\text{tot}}_{rr} = (\tau_r + 4\tau_{r+1} + \tau_{r+2}) \\ \text{for } r = 1,.., N \end{gathered} \tag{67}$$

The other values of $k^{s*\,tot}_{ij}$ are obtained by symmetry relations, or are equal to zero.

3. APPLICATION

The present section is devoted to the presentation and discussions of the numerical results obtained by application of the theory established above to a Cracked Simply Supported Beam Resting on Elastic Foundation (CSSBREF) in both the linear and nonlinear cases. The beam examined has the following properties: length L=10m, Young's modulus E=2.068x10^{11}N/m², density =7850kg/m^3, the cross section width b=0.25m and height h=0.25m and soil parameter λ =10.

3.1. Linear case

The CSSBREF nondimensional stiffness matrix $\left[\mathbf{K}_N^{SS}*\right]$ can be expressed as:

$$\left[\mathbf{K}_N^{SS}\right] = \frac{(N+1)^3 EI}{L^3}\left[\mathbf{K}_N^{SS}*\right] \tag{68}$$

with :

$$\left[\mathbf{K}_N^{SS}*\right] = \begin{bmatrix} 5+\alpha & -4 & 1 & 0 & . & . & . & . & 0 \\ -4 & 6+\alpha & -4 & 1 & 0 & . & . & . & . \\ 1 & -4 & 5+\alpha+\tau_r & . & \tau_r & . & . & . & . \\ 0 & 1 & . & 2+\alpha+4\tau_r & -2-2\tau_r & 1 & . & . & . \\ . & 0 & \tau_r & -2-2\tau_r & 5+\alpha+\tau_r & -4 & . & 0 & . \\ . & . & . & 1 & -4 & 6+\alpha & . & 1 & 0 \\ . & . & . & . & . & . & . & -4 & 1 \\ . & . & . & . & 0 & 1 & -4 & 6+\alpha & -4 \\ 0 & . & . & . & . & 0 & 1 & -4 & 5+\alpha \end{bmatrix} \tag{69}$$

The mass matrix is equal to the identity matrix, and equation (12) corresponding to the beam linear vibrations reduced to:

$$\left[\mathbf{K}_N^{SS}*\right]\{\mathbf{y}\} - \left(\omega_{discr}^{SS}*\right)^2[\mathbf{I}]\{\mathbf{y}\} = 0 \tag{70}$$

where $\omega_{discr}^{SS}*$ is the dimensionless linear frequency of the discrete system presenting the CSSBREF. The solution to equation (70) leads to N eigenvalues β_i, i=1 to N, associated to N vibration modes and related to N frequencies $\omega_{discr}^{SS}*$, given by:

$$\omega_i^{SS}{}_{discr}* = (N+1)^2\sqrt{\beta_i} \tag{71}$$

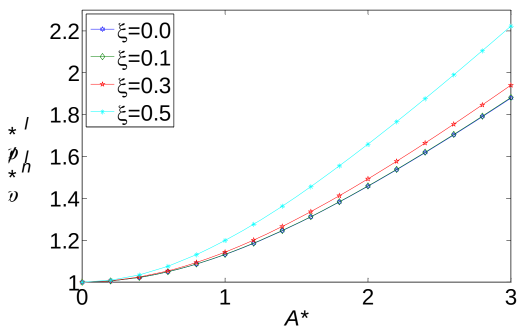
In Table 1, the nondimensional fundamental frequencies of CSSBREF, corresponding to various values of the crack position and depth, are presented and compared to those of Shin et al. [13]. The results are given, for N=10; 20; 30, showing a normal process of convergence. Table 1 shows, as may be expected, that the frequencies obtained by the discrete model for cracked beams are in all cases lower compared to the frequency of the uncracked beam. The results agree well with those of [13] since the overall maximum difference does not exceed 1% for N>20. It may also be noticed that the fundamental frequencies are more affected by cracks located around the middle point of the beam while the second frequencies are more affected by cracks located around x=L/4, which may be expected since these locations correspond in each case to the antinodes of the corresponding modes.

3.2. Nonlinear case

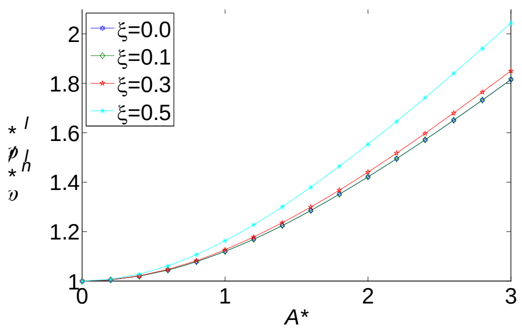
To analyze the results in the nonlinear case, comparing the ratio of the nonlinear frequency parameter ($\omega_{nl}^{ss}*_{discr}/\omega_l^{ss}*_{discr}$) may be one of the best indicators of the nonlinearity type and acuity. The number of masses considered in the calculation is (N=30), because of the good approximation obtained for this degree of discretization [18]. Table 2 gives the nonlinear frequencies for increasing values of the vibration amplitude and the soil parameter λ. The parameters of the beam considered are the same as those taken in the linear case.

In Table 2 and Fig. 3, nonlinear vibrations of un-cracked beams (ξ =0.0) and cracked beams with increasing crack depth up to (ξ =0.5) are examined in the neighbourhood of the fundamental mode for various values of the soil rigidity λ. The results show that increasing the vibration amplitude induces in all cases an increase in the nonlinear frequency parameter ($\omega_{nl}^{ss}{*}_{discr} / \omega_{l}^{ss}{*}_{discr}$) according to the expected hardening type nonlinear behaviour. On the other hand, increasing the crack depth induces an increase in the non-linearity effect, which may be explained by equation (58) in which the a^2 coefficient is B/K which increases

(a)



(b)



(c)

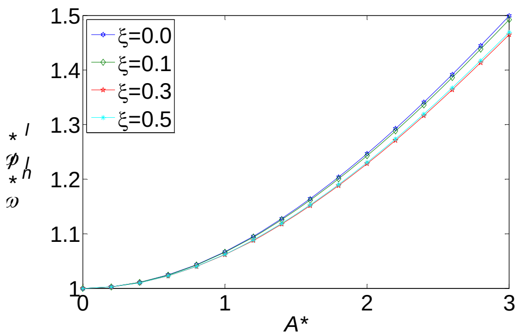


Fig. 3. Comparison of the non-linear frequency ratio of the first non-linear mode shape of a CSSBREF with crack at the position $x=L/2$: (a) λ=0, (b) λ=10, (c) λ=100,

when the crack depth increases. Also, the nonlinear effect in the neighbourhood of a given mode depends on the crack position according the analysis made above for the linear frequencies (section 3.1).

4. CONCLUSION

A theoretical general discrete model for the non-linear vibration of cracked beams resting on elastic foundations has been established. This model is an extension of the theories previously developed for uncracked beams [18] to beams with an edge crack located at different positions. First the model has been validated for a CSSBREF in the linear case via comparison with the literature [13]. In the nonlinear case, the applications carried out for a CSSBREF, enable one to assess the effects of the large vibration amplitudes, the crack depth and the soil parameter on the amplitude dependent non-linear frequencies for the first three modes.

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Received 2017-04-14
Accepted 2017-06-06
Available online 2017-08-10

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